

The Gödel and the Splitting Translations

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Abstract. When the new research area of logic programming and non-monotonic reasoning emerged at the end of the 1980s, it focused notably on the study of mathematical relations between different non-monotonic formalisms, especially between the semantics of stable models and various non-monotonic modal logics. Given the many and varied embeddings of stable models into systems of modal logic, the modal interpretation of logic programming connectives and rules became the dominant view until well into the new century. Recently, modal interpretations are once again receiving attention in the context of hybrid theories that combine reasoning with non-monotonic rules and ontologies or external knowledge bases.

In this talk I explain how familiar embeddings of stable models into modal logics can be seen as special cases of two translations that are very well-known in non-classical logic. They are, first, the translation used by Gödel in 1933 to embed Heyting's intuitionistic logic H into a modal provability logic equivalent to Lewis's $S4$; second, the splitting translation, known since the mid-1970s, that allows one to embed extensions of $S4$ into extensions of the non-reflexive logic, $K4$. By composing the two translations one can obtain (Goldblatt, 1978) an adequate provability interpretation of H within the Goedel-Loeb logic GL , the system shown by Solovay (1976) to capture precisely the provability predicate of Peano Arithmetic. These two translations and their composition not only apply to monotonic logics extending H and $S4$, they also apply in several relevant cases to non-monotonic logics built upon such extensions, including equilibrium logic, non-monotonic $S4F$ and autoepistemic logic. The embeddings obtained are not merely faithful and modular, they are based on fully recursive translations applicable to arbitrary logical formulas. Besides providing a uniform picture of some older results in LPNMR, the translations yield a perspective from which some new logics of belief emerge in a natural way.

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1 Introduction

When the new area of logic programming and non-monotonic reasoning emerged towards the end of the 1980s much attention was paid the semantics of stable models and its relation to systems of non-monotonic modal logic. Several different formal embedding relations of stable model semantics into modal logics were discovered by Schwarz, Gelfond, Lifschitz, Marek, Truszczyński, Chen and others. In this manner the modal-epistemic interpretation of logic programming rules was born and became a dominant view in the 1990s and even well into the new Century.

These embedding relations are interesting for several reasons. For one thing, it is well-known that provability in formal systems can be given a modal interpretation and so the new negation as failure-to-prove in logic programming might perhaps turn out to be related in a natural way to modal concepts of this type. Secondly, logic programs had a special kind of syntax based on *rules*, while their modal translations were sets of ordinary logical formulas. So the embeddings seemingly related quite different kinds of syntactic objects. Third, while the translations were not arbitrary (cf. provability concepts of modality), they were largely *ad hoc*. That is to say, they were discovered and found to work in a practical sense, but were not based on a single underlying, systematic methodology. Indeed it seemed that as one moved from simpler to more complex types of program rules, their modal translations actually changed. A comprehensive theory to explain this was lacking. Moreover, while the successful embedding relations were *modular*, they were generally defined on complete rules, one at a time, rather than being built up recursively from the rule components.

The aim of this paper is to revisit the modal and epistemic interpretation of logic programs under stable model semantics and try to supply an overarching theory that links these different translations and explains in a certain sense why they work. What I shall try to explain is how all of the familiar embeddings of stable models into modal logics can be seen as special cases of two translations that are very well-known in non-classical logic. They are, first, the translation used by Goedel in 1933 to embed Heyting's intuitionistic logic H into a modal provability logic equivalent to Lewis's $S4$; second, the splitting translation, known since the mid-1970s, that allows one to embed extensions of $S4$ into extensions of the non-reflexive logic, $wK4$.

I hope this approach will be of historical as well as didactical interest. But it may also prove useful beyond this. Recently, modal interpretations of logic programs are once again receiving attention in the context of hybrid theories that combine reasoning with non-monotonic rules and ontologies or external knowledge bases, often considered in the framework of the so-called Semantic Web. It seems plausible that at least some of these interpretations – since they involve translating answer set programs – will also be closely related to the Goedel and the splitting translations. Another feature of interest is that when we build up a picture of different logics, monotonic and non-monotonic, related by these translations, we actually discover some gaps in the picture that can be filled by 'new' logics that may be of interest in their own right. I will give an example of this in due course.

1.1 Plan of the paper

The rest of the paper will be organised around commutative diagrams of different logics and their inter-relations. We'll start with the basic case that was dealt with by Gödel and by Tarski and McKinsey: that of intuitionistic logic and modal $S4$. As we proceed we will extend this diagram to include more logics, adding also the splitting translation. Eventually we will have a set of base logics, their extensions and also their non-monotonic versions. Once this diagram is complete we will return to some of the embeddings of stable model semantics that arose in the early years of logic programming and non-monotonic reasoning. We'll see how to derive those embeddings using the two basic translations that we started with. We'll conclude by discussing some possible extensions of this approach as well as some of its limitations.

2 Logical preliminaries

I shall assume familiarity with basic intuitionistic and modal propositional logic. What follows is a brief summary that serves mainly to fix notation and terminology. For more details the reader should consult the numerous introductions that can be found in handbook chapters and several textbooks, eg. *Syntax*. We consider propositional languages equipped with an infinite set $Prop$ of propositional letters. In both cases the symbols $\vee, \wedge, \neg, \rightarrow$ denote the propositional connectives disjunction, conjunction, negation and implication, respectively.¹ We consider normal modal logics containing the additional necessity operator, \Box . As usual $\Diamond\varphi$ denotes the proposition $\neg\Box\neg\varphi$. The axioms are all classical tautologies plus a selection of the axioms listed below. Rules of inference are: modus ponens, substitution and necessitation.

$$\begin{aligned} K : & \Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q) \\ 4 : & \Box p \rightarrow \Box\Box p \\ w4 : & \Box p \wedge p \rightarrow \Box\Box p \\ 5 : & \neg L\neg Lp \rightarrow Lp \\ W5 : & \neg L\neg Lp \rightarrow (p \rightarrow Lp) \\ D : & \neg Lp \vee \neg L\neg p \\ T : & \Box p \rightarrow p \\ F : & (p \wedge MLq) \rightarrow L(Mp \vee q) \\ f : & p \wedge \Diamond(q \wedge \Box\neg p) \rightarrow \Box(q \vee \Diamond q) \end{aligned}$$

Well-known examples of modal logics we shall deal with are $S4$, containing the axioms K, T and 4 , the logic $S5$ which extends $S4$ by adding axiom 5 , and $KD45$ comprising precisely $K, D, 4$ and 5 .

Intuitionistic propositional logic is determined by the usual intuitionistic axioms and the rule modus ponens (see eg [8]). We denote this calculus by \mathcal{H} for Heyting. *Super-intuitionistic logics* are obtained by adding further axioms to \mathcal{H} . Here we consider one

¹ Context should make it clear whether we are dealing with an intuitionistic connective or a modal one.

such logic in particular, the logic of *here-and-there*, denoted by HT . This is obtained by adding to \mathcal{H} the axiom of Hosoi [?]:

$$\alpha \vee (\neg\beta \vee (\alpha \rightarrow \beta))$$

HT is the strongest extension of \mathcal{H} that is properly contained in classical logic. It can be equivalently presented as a 3-valued logic and is sometimes known as Gödel's 3-valued logic.

Semantics. An (intuitionistic, Kripke) *frame* is a pair $\mathcal{F} = \langle W, \leq \rangle$, where W is a non-empty set and \leq a partial ordering on W . A Kripke *model* \mathcal{M} is a frame together with a valuation $V : Prop \times W \rightarrow \{0, 1\}$. V is extended recursively to all formulas by the usual truth conditions for intuitionistic Kripke semantics [8]. In particular we have the following clause for implication

$$V(\alpha \rightarrow \beta, w) = 1 \text{ iff } V(\beta, w') = 1 \text{ whenever } V(\alpha, w') = 1, \text{ for all } w' \geq w \quad (1)$$

Given a model $\mathcal{M} = \langle W, \leq, V \rangle$ we also write $\mathcal{M}, w \models \varphi$ to denote $V(\varphi, w) = 1$. Similarly, an intuitionistic formula φ is said to be *true* in a model \mathcal{M} , denoted by $\mathcal{M} \models \varphi$, if $\mathcal{M}, w \models \varphi$ for all $w \in W$ (likewise for sets of formulas). A formula is valid on a class of frames if it is true in all models based on those frames. Heyting's calculus \mathcal{H} is sound and complete with respect to this semantics, in particular the theorems of \mathcal{H} are precisely the formulas valid on the class of all frames. For any super-intuitionistic logic \mathcal{I} based on some class \mathcal{K} of frames, we can define a concept of (local) *consequence* as follows. $\Sigma \models_{\mathcal{I}} \varphi$ if for any model $\mathcal{M} = \langle W, \leq, V \rangle$ where $\langle W, \leq \rangle$ is a frame in \mathcal{K} , and any point $x \in W$, $\mathcal{M}, x \models \varphi$ whenever $\mathcal{M}, x \models \Sigma$.

The logic HT is based on rooted frames having just two elements, h and t , with $h \leq t$. It is often convenient to represent an HT -model $\langle \{h, t\}, \leq, V \rangle$ as an ordered pair of sets of atoms, $\langle H, T \rangle$, where $H = \{p \in Prop : V(p, h) = 1\}$ and similarly $T = \{p \in Prop : V(p, t) = 1\}$. By the usual persistence requirement it follows that $H \subseteq T$. Truth, validity and consequence for HT are defined in the usual way.

Similarly a *modal frame* $\mathcal{F} = \langle W, R \rangle$ comprises a set W together with a binary relation R on W . Again, modal models are frames equipped with a valuation $V : Prop \times W \rightarrow \{0, 1\}$, where V is extended recursively to all formulas by the usual truth conditions. In particular we have the standard condition for necessity:

$$V(L\varphi, w) = 1 \text{ iff } V(\varphi, w') = 1 \text{ for all } w' \text{ s.t. } wRw'. \quad (2)$$

For modal logics the notions of truth, validity and consequence are defined analogously to the intuitionistic cases described above and we adopt similar notational conventions.

2.1 The Gödel translation

In his 1933 paper Gödel [9] provided an interpretation of \mathcal{H} into a logical system equivalent to $S4$. His original translation of intuitionistic into modal formulas, denoted here

by τ , is defined as follows:

$$\begin{aligned}\tau(p) &= Lp \\ \tau(\varphi \wedge \psi) &= \tau(\varphi) \wedge \tau(\psi) \\ \tau(\varphi \vee \psi) &= \tau(\varphi) \vee \tau(\psi) \\ \tau(\varphi \rightarrow \psi) &= L(\tau(\varphi) \rightarrow \tau(\psi)) \\ \tau(\neg\varphi) &= L\neg\tau(\varphi)\end{aligned}$$

If we add $\tau(\perp) = \perp$, then the translation of $\neg\varphi$ becomes derivable via the definition $\neg\varphi := \varphi \rightarrow \perp$. Gödel also noted that variations on this translation will also work; a frequently used variant is the translation that places an ‘ L ’ before every subformula.² Gödel essentially established that for an intuitionistic formula φ ,

$$\vdash_{\mathcal{H}} \varphi \Rightarrow \vdash_{S4} \tau(\varphi), \quad (3)$$

(where \vdash with subscripts denotes theoremhood) and he conjectured that the converse relation also holds, in other words that τ is a faithful interpretation or *embedding*. This conjecture was later proved by McKinsey and Tarski in their 1948 paper [36].

Following [16], McKinsey and Tarski [34–36] also laid the foundations for the algebraic and topological study of intuitionistic and modal logics. The basic idea, recalled and developed in a recent paper by Leo Esakia [7], is that from an arbitrary topological space X we can generate three different algebraic structures each giving rise to different logical systems.³ In particular, by considering the algebra of open sets, $Op(X)$, one is led to the well-known Heyting algebra that also forms a semantical basis for intuitionistic logic. On the other hand by considering the closure algebra, $(\mathcal{P}(X), \mathbf{c})$ one is led to the modal system $S4$.⁴ In [36] McKinsey and Tarski proved the topological completeness of $S4$ where \Diamond is interpreted as the closure operator \mathbf{c} .

By the 1970s logicians began the systematic study of relations between (the lattice of) extensions of \mathcal{H} and (the lattice of) normal extensions of $S4$. Also, since (as Gödel [9] already observed) $S4$ -necessity does not correspond to provability in a formal system, interest grew in finding modal systems that directly model provability concepts. Around 1976 Esakia introduced the expression “modal companion” of a super-intuitionistic logic \mathcal{I} to refer to those modal systems into which \mathcal{I} can be embedded via the Gödel translation.⁵ He also established in [5, 6] that \mathcal{H} has a strongest modal companion, the so-called Grzegorczyk logic, Grz , already known to be a modal companion of \mathcal{H} , [10]. Grz is the extension of $S4$ obtained by adding the schema

$$L(L(p \rightarrow Lp) \rightarrow p) \rightarrow p.$$

² The version of the Gödel translation given here by τ is actually the one used by McKinsey and Tarski [36] and is commonly found in the literature.

³ For the basic notions of topology see eg [12] or any appropriate textbook.

⁴ Recall that a Heyting algebra $(\mathbf{H}, \vee, \wedge, \rightarrow, \perp)$ is a distributive lattice with smallest element \perp containing a binary operation \rightarrow such that $x \leq a \rightarrow b$ iff $a \wedge x \leq b$. $(\mathbf{B}, \vee, \wedge, -, \mathbf{c})$ is a closure algebra if $(\mathbf{B}, \vee, \wedge, -)$ is a Boolean algebra and \mathbf{c} is a closure operator satisfying: $a \leq \mathbf{c}a$, $\mathbf{c}ca = \mathbf{c}a$, $\mathbf{c}(a \vee b) = \mathbf{c}a \vee \mathbf{c}b$, $\mathbf{c}\perp = \perp$.

⁵ The expression is translated from the Russian ????????? ????????? .

Esakia's result is part of a more general pattern: the Blok-Esakia Theorem [1, 5] establishes an isomorphism between the lattice of intermediate logics and the lattice of all normal extensions of Grz .

2.2 The splitting translation

In the same year, 1976, Solovay [40] proved the arithmetical completeness of the so-called Gödel-Löb modal logic, GL , establishing a correspondence between derivability in GL and provability in the formal system of Peano Arithmetic, PA. GL results from $K4$ by adding the schema

$$L(Lp \rightarrow p) \rightarrow Lp.$$

At this point it is appropriate to turn to the splitting translation that will be denoted here by a superscript operator '+'. This is a translation from modal formulas to modal formulas that replaces each occurrence of L by L^+ where $L^+\varphi$ abbreviates

$$\varphi \wedge L\varphi,$$

other formulas being left unchanged. It appears that the splitting translation was independently discovered by several authors and its first main application is usually attributed to Kuznetsov and Muravitsky [29], Goldblatt [26] and Boolos [2]. They established the following embedding of the reflexive logic Grz into the non-reflexive GL .

$$\vdash_{Grz} p \Leftrightarrow \vdash_{GL} p^+. \quad (4)$$

Not surprisingly, as Goldblatt [26] showed, one can form the composition τ^+ of τ with $^+$ and combine (3) and (4) to yield

$$\vdash_{\mathcal{H}} p \Leftrightarrow \vdash_{GL} \tau^+(p), \quad (5)$$

(setting $\tau^+(p) = (\tau(p))^+$) which using Solovay's result yields a provability interpretation of intuitionistic logic.

If we depict these relations in diagrammatic form, from the simple picture

$$\mathcal{H} \xrightarrow{\tau} S4$$

we have now reached the following situation

$$\begin{array}{ccc} & & Grz \xrightarrow{+} GL \\ & \nearrow \tau & \uparrow \\ \mathcal{H} & \xrightarrow{\tau} & S4 \end{array}$$

The splitting translation has two very natural interpretations. One of them is topological. A third path from topology to logic is via what are known as *derivative algebras*, $(\mathcal{P}(X), der)$. These are Boolean algebras with a unary operation der representing topological derivation: if A is a subset of X then $der(A)$ is the set of all accumulation

or limit points of A . Under this topological reading of modal logics Mp is interpreted as derivation. Since in topology closure is definable in terms of derivation, viz. for a point set A , $c(A) = A \cup der(A)$, we obtain the following modal connection: Mp is identified with $p \vee Mp$. But this is precisely the splitting translation when applied to M , namely $M^+p = p \vee Mp$.

In [37] Esakia showed that the derivative algebra $(P(X), der)$ gives rise to the modal logic $wK4$, a slightly weaker version of the logic $K4$, that was first studied from a topological point of view in [4] (see [7] for a detailed overview). $wK4$ is obtained by adding the axiom schema $w4$ to K . It is known to be the weakest normal extension of K into which $S4$ embeds via the splitting translation. This means that we can now complete our previous picture by adding in a missing logic, namely the logic of all topological spaces interpreting M as the derivative operator:⁶

$$\begin{array}{ccccc}
 & & Grz & \xrightarrow{+} & GL \\
 & \nearrow \tau & \uparrow & & \uparrow \\
 \mathcal{H} & \xrightarrow{\tau} & S4 & \xrightarrow{+} & wK4
 \end{array}$$

The logics on the right of our picture are interesting from another perspective as well. While extensions of $S4$ may be considered good candidates for modelling epistemic reasoning about knowledge, extension of $wK4$ are candidates to form logics of belief. In particular they may not have the strong T axiom $Lp \rightarrow p$. A standard doxastic logic, $KD45$, is one such extension of $wK4$. From this epistemic perspective the splitting translation is also very natural. Interpreting Lp in extensions of $S4$ as “the agent knows p ” and in extensions of $wK4$ as “the agent believes p ”, then knowledge is interpreted via the splitting translation as ‘truth plus belief’.

3 Monotonic embeddings

Up to this point we have remarked how the systems \mathcal{H} , $S4$ and $wK4$ represent three distinct paths to logic via topology, each logic captured in its own way by the class of all topological spaces. We also saw how the Gödel translation τ embeds \mathcal{H} into $S4$ and how $wK4$ is the least logic into which $S4$ embeds via the splitting translation, $+$.

Let us now turn to logics obtained from *minimal* topological spaces. A topological space is minimal if it has only three open sets. Considering the algebra of open sets we arrive at the 3-element Heyting algebra which captures precisely the logic HT of here-and-there. Taking as a starting point the closure algebra $(\mathcal{P}(X), c)$ we obtain the modal logic $S4F$ that adds the F axiom schema to $S4$. The semantics of $S4F$ was first studied by Segerberg [15]. However in [18, 19] Schwarz and Truszczyński proposed a new approach to minimal knowledge and suggested that this concept is precisely captured by non-monotonic $S4F$. In [19] it was shown that non-monotonic $S4F$ captures, under some intuitive encodings, several important approaches to knowledge representation. They include disjunctive logic programming under answer set semantics [?],

⁶ Vertical arrows always denote extensions.

(disjunctive) default logic [23], [25], the logic of grounded knowledge [27], the logic of minimal belief and negation as failure [28] and the logic of minimal knowledge and belief [19]. Recently, Truszczyński [31] and Cabalar [32] have revived the study of $S4F$ in the context of a general approach to default reasoning.

Frames for $S4F$ have the form depicted in Figure 1 where W_1 and W_2 are clusters,

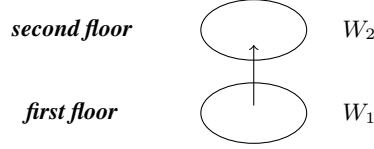


Fig. 1.

all points are reflexive and every point in W_2 is accessible from every point in W_1 . We call W_1, W_2 respectively the *first* and the *second floor* of the model. The former may be empty but the latter not. In these frames the accessibility relation is of course a preorder or quasi-order.

If the possibility operator ' M ' is construed as topological derivation then, as Pearce and Uribe [38] recently showed, as the logic corresponding to the class of minimal topological spaces one obtains $wK4f$, the system extending $wK4$ by adding the schema f . [38] also establishes soundness and completeness of $wK4f$ with respect to a class of modal frames. The frames for $wK4f$ are similar to those of $S4F$ except that we drop

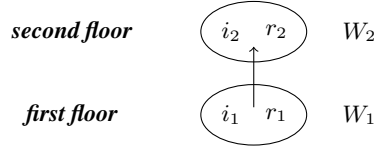


Fig. 2.

the condition of reflexivity on frames: in Fig. 2 some points in W_1, W_2 may now be irreflexive (where i and r label this difference). Accessibility for these frames is only weakly-transitive [38].

Therefore by considering logics extending H , $S4$ and $wK4$ based on *minimal* topological spaces, we obtain respectively the systems HT , $S4F$ and $wK4f$. So our picture now looks like this:

$$\begin{array}{ccccc}
\text{min-topologies} & HT & S4F & wK4f \\
& \uparrow & \uparrow & \uparrow \\
\text{topologies} & H & \xrightarrow{\tau} S4 & \xrightarrow{+} wK4
\end{array}$$

Our main objective is to consider non-monotonic versions of the logics HT , $S4F$ and $wK4f$ and to show how the first of these, equilibrium logic, in the diagram below EL , embeds into the other two. In diagrammatic form we want to show

$$\text{nonmonotonic} \quad EL \xrightarrow{\tau} S4F^* \xrightarrow{+} wK4f^*$$

where the starred version of a modal logic indicates its non-monotonic version. Rather than attempt to do this directly, we will build on the embedding relations that hold at the monotonic level and then show how these can be lifted to the non-monotonic cases. So we will deal with this picture:

$$\begin{array}{ccccc}
\text{nonmonotonic} & EL & \xrightarrow{\tau} S4F^* & \xrightarrow{+} wK4f^* \\
& \uparrow & \uparrow & \uparrow \\
\text{min-topologies} & HT & \xrightarrow{\tau} S4F & \xrightarrow{+} wK4f
\end{array}$$

Actually the embeddings at the bottom of this picture can be established by standard methods. But since these are quite general and applicable to many logics, it may be a useful exercise to review them here. We'll omit detailed proofs but give the main ideas and lemmas needed.

Let $\mathcal{F} = \langle W, R \rangle$ be a modal frame. We define the reflexivization of \mathcal{F} to be the frame $\mathcal{F}^r = \langle W, R^r \rangle$ where

$$xR^ry \text{ iff } x = y \text{ or } xRy. \quad (6)$$

It is easy to see that if \mathcal{F} is a one-step, weakly transitive frame for $wK4f$ then \mathcal{F}^r is transitive and a frame for $S4F$. Given a model $\mathcal{M} = (\mathcal{F}, V)$, \mathcal{M}^r is the model (\mathcal{F}^r, V) .⁷

Lemma 1. *For every model \mathcal{M} , every point x in \mathcal{M} and every formula φ ,*

$$\mathcal{M}, x \models \varphi^+ \text{ iff } \mathcal{M}^r \models \varphi.$$

The proof is by induction on φ . From this “reflexivization” lemma we can deduce that for every frame \mathcal{F} and formula φ ,

$$\mathcal{F} \models \varphi^+ \text{ iff } \mathcal{F}^r \models \varphi. \quad (7)$$

⁷ The following few steps and results seem to be standard and quite well-known in modal logic, so I am not sure how to give precise references. However, I learnt the method from Misha Zakharyashev who several years ago supplied me with a detailed account and proofs. Moreover a good reference for this topic is the paper by Chagrov and Zakharyashev.

On this basis we can relate the frames for $wK4f$ to those for $S4F$. Now consider the Goedel translation, τ . Corresponding to this there is a semantic map that relates quasi-ordered modal frames to partially order intuitionistic frames. We denote this map by ρ and define it as follows. Let \mathcal{F} be a quasi-ordered frame and $\mathcal{M} = \langle \mathcal{F}, V \rangle$ a model based on it. The *skeleton* $\rho\mathcal{F}$ of \mathcal{F} is obtained from \mathcal{F} by collapsing clusters to single points, resulting in a partially-ordered frame. The corresponding model $\rho\mathcal{M} = \langle \rho\mathcal{F}, \rho V \rangle$, called the *skeleton* of \mathcal{M} , is defined by the intuitionistic valuation ρV which for each variable p , sets

$$\rho V(p) = \{C(x) : \mathcal{M}, x \models \Box p\}, \quad (8)$$

where $C(x)$ is the cluster generated by x , ie it is a single point of $\rho\mathcal{F}$.

The following “skeleton lemma” is another well-known result that is proved by induction on formulas φ .

Lemma 2. *Let \mathcal{M} be a modal model based on a quasi-ordered frame. Then for every point x in \mathcal{M} and every intuitionistic formula φ ,*

$$\rho\mathcal{M}, C(x) \models \varphi \text{ iff } \mathcal{M}, x \models \tau(\varphi).$$

Likewise we can deduce that for every quasi-ordered frame \mathcal{F} and intuitionistic formula φ we have

$$\rho\mathcal{F} \models \varphi \text{ iff } \mathcal{F} \models \tau(\varphi). \quad (9)$$

We define the translation τ^+ by setting $\tau^+(\varphi) = (\tau(\varphi))^+$. Then our embedding theorem can be stated as follows.

Theorem 1. *The translation τ^+ is an embedding of HT into $wK4f$, ie for any intuitionistic formula φ ,*

$$\models_{HT} \varphi \text{ iff } \models_{wK4f} \tau^+(\varphi).$$

Proof. Suppose that $\not\models_{wK4f} \tau^+(\varphi)$. Then there is a $wK4f$ frame \mathcal{F} such that $\mathcal{F} \not\models \tau^+(\varphi)$. Hence $\mathcal{F}^r \not\models \tau(\varphi)$ and so $\rho(\mathcal{F}^r) \not\models \varphi$. However $\rho(\mathcal{F}^r)$ is isomorphic to the HT -frame \mathcal{H} , meaning that $\not\models_{HT} \varphi$. For the other direction, if $\not\models_{HT} \varphi$ then $\mathcal{H} \not\models \varphi$. Hence $\mathcal{F}^r \not\models \tau(\varphi)$ and so $\mathcal{F} \not\models \tau^+(\varphi)$. \square

Evidently we can decompose τ^+ to deduce that τ embeds HT into $S4F$ and $+$ embeds $S4F$ into $wK4f$.

4 Non-monotonic embeddings

Now we turn to the non-monotonic versions of these logics. For the case of a normal modal logic \mathcal{S} the standard way to define its non-monotonic version that we’ll denote by \mathcal{S}^* is via a fixpoint condition that defines the so-called \mathcal{S} -expansions.

Definition 1. *Let \mathcal{S} be a normal modal logic with consequence relation $Cn_{\mathcal{S}}$. Let I be a set of modal formulas. A set of formulas E is said to be an \mathcal{S} -expansion of I iff*

$$E = Cn_{\mathcal{S}}(I \cup \{\neg\Box\varphi : \varphi \notin E\}).$$

Then \mathcal{S}^* is the non-monotonic logic determined by truth in all \mathcal{S} -expansions, ie. we can define the non-monotonic entailment relation by $I \vdash_{\mathcal{S}^*} \varphi$ iff $\varphi \in E$ for each \mathcal{S} -expansion E of I .

Similarly, the logic HT has a natural non-monotonic extension that we call *equilibrium logic*, EL . This can also be captured by a similar fixpoint condition.

Definition 2. Let X be a set of intuitionistic formulas. A set of formulas C is said to be a completion of X iff

$$C = Cn_{HT}(X \cup \{\neg\varphi : \varphi \notin C\}).$$

Again equilibrium logic is determined by truth in all completions.

For proving results about these logics it is often easier to work with equivalent characterisations using minimal or preferred models. For the case of EL this was the original definition of the logic in terms of special kinds of minimal HT -models called *equilibrium models*.

Definition 3. Among HT -models we define the order \trianglelefteq by: $\langle H, T \rangle \trianglelefteq \langle H', T' \rangle$ if $T = T'$ and $H \subseteq H'$. If the subset relation is strict, we write ' \triangleleft '.

Definition 4. Let X be a set of intuitionistic formulas and $\mathcal{M} = \langle H, T \rangle$ a model of X . \mathcal{M} is said to be an equilibrium model of X if it is minimal under \trianglelefteq among models of X , and it is total.

Equilibrium models of theories correspond to their completions, so we can define inference by

$$X \vdash \varphi \text{ iff } \mathcal{M} \models \varphi \text{ for each equilibrium model } \mathcal{M} \text{ of } X. \quad (10)$$

Certain kinds of non-monotonic modal logics are captured by a concept of minimal model that was introduced by Schwarz [39].

Definition 5. Let $\mathcal{N} = (N, S, U)$ be a two-floor $S4F$ model as depicted in Figure 1. We say that \mathcal{N} is preferred over an $S5$ -model $\mathcal{M} = (W, R, V)$ if:

- a) There is a propositional formula ψ such that $\mathcal{M} \models \psi$ and $\mathcal{N} \not\models \psi$,
- b) (W, R) is the second floor of (N, S) and V equals to the restriction of U to the second floor. Briefly, \mathcal{M} is the model which is obtained by deleting the first floor in \mathcal{N} .

From this one obtains the notion of minimal model that is central for the semantics of non-monotonic modal logics.

Definition 6. An $S5$ -model $\mathcal{M} = (W, R, V)$ is called a \mathcal{K} -minimal model for the set of formulas I if $\mathcal{M} \models I$ and for every preferred model $\mathcal{N} \in \mathcal{K}$ we have $\mathcal{N} \not\models I$.

Schwarz showed that for logics \mathcal{S} such as $S4F$ that are cluster-closed, minimal models correspond to \mathcal{S} -expansions and so characterise the logic. For the case of $wK4f$ we cannot directly apply Schwarz's Theorem since the class \mathcal{K} which characterises $wK4f$ is not cluster-closed. In particular some two-floor models in \mathcal{K} may not have a cluster as a maximum. On the other hand every model in \mathcal{K} has a maximal *weak-cluster*

(that is a cluster where irreflexive points are allowed or more precisely it is a rooted, symmetric, weakly-transitive frame).

It turns out that Schwarz's Theorem can be extended to the weak-cluster closed class \mathcal{K} , so accordingly we extend Definitions 5 and 6 to include this class. We then obtain the following recent result from [38].

Theorem 2. *Let $\mathcal{M} = (W, R, V)$ be an $S5$ -model, and $T = \{\varphi : \mathcal{M} \Vdash \varphi\}$. Then T is an $wK4f$ -expansion of I if and only if \mathcal{M} is a \mathcal{K} -minimal model of I .*

We are now ready to prove that τ^+ embeds equilibrium logic into non-monotonic $wK4f$. Although we could prove this in one step, it is more convenient to deal with the τ and the $^+$ embeddings separately and then compose them at the end. So, first let's show how the relexivization map r preserves the property of being a minimal model.

Lemma 3. *Let I be a set of modal formulas. \mathcal{M} is a minimal $wK4f$ -model of I^+ if and only if \mathcal{M}^r is a minimal $S4F$ -model of I .*

Proof. For the 'if' direction suppose that r is the mapping from $wK4f$ models to $S4F$ models that makes irreflexive points reflexive. Let I be a theory in $S4F$ and consider any minimal model \mathcal{M} of I^+ . Since \mathcal{M} is an $S5$ -model, $\mathcal{M}^r = \mathcal{M}$. Suppose it is not a minimal model of I . Then there is a preferred model $\mathcal{M}' < \mathcal{M}^r$ such that $\mathcal{M}' \models I$ but there is a propositional formula α such that \mathcal{M}^r verifies α but \mathcal{M}' does not. In particular \mathcal{M}' is a 2-floor model that we can represent as the pair $(\mathcal{N}, \mathcal{M}')$, where \mathcal{N} is the first-floor cluster. Now consider $(\mathcal{N}, \mathcal{M})$ as a $wK4f$ model. Evidently $(\mathcal{N}, \mathcal{M})^r = \mathcal{M}'$. So, by the reflexivization lemma, for any formula φ ,

$$\mathcal{M}' \models \varphi \text{ iff } (\mathcal{N}, \mathcal{M})^r \models \varphi \text{ iff } (\mathcal{N}, \mathcal{M}) \models \varphi^+.$$

Now since α is an (objective), propositional formula, $\alpha^+ = \alpha$ and so $(\mathcal{N}, \mathcal{M}) \not\models \alpha^+$. On the other hand, $(\mathcal{N}, \mathcal{M}) \models I^+$. But this contradicts the assumption that \mathcal{M} is a minimal model of I^+ . It follows that \mathcal{M}^r is a minimal model of I .

For the 'only if' direction the argument is quite similar, applying the Reflexivization Lemma. It is left to the reader. \square

We have established one half of the embedding. Now let us turn to the semantic mapping between $S4F$ -models and HT models and show that it preserves the property of being a minimal model.

Lemma 4. *Let X be a set of HT -formulas. \mathcal{M} is a minimal model of $\tau(X)$ if and only if $\rho\mathcal{M}$ is an equilibrium model of X .*

Proof. For the 'if' direction, suppose that \mathcal{M} is a minimal model of $\tau(X)$. Recall that \mathcal{M} is an $S5$ -model and $\rho V(p) = \{C(x) : \mathcal{M}, x \models \Box p\}$. Now suppose for the contradiction that $\rho\mathcal{M}^r$ is not an equilibrium model of X . Then there is an HT -model \mathcal{N} of X such that $\mathcal{N} \trianglelefteq \rho\mathcal{M}$. This means that for some propositional variable p we have $\mathcal{N}, t \models p$ and $\mathcal{N}, h \not\models p$. Suppose \mathcal{N} has the form $\mathcal{N} = \langle \rho\mathcal{F}', U \rangle$ where $\mathcal{F}' = \langle W, R \rangle$ is an $S4F$ frame. By setting for each p

$$V'(p) = \{x \in W : \mathcal{N}, C(x) \models p\}$$

one obtains a modal ($S4F$) model $\langle \mathcal{F}', V' \rangle$ whose skeleton is (isomorphic to) \mathcal{N} . By construction $\langle \mathcal{F}', V' \rangle$ has two clusters. Its second floor is equivalent to \mathcal{M} and verifies $\Box p$ and hence p , while its first floor does not verify p . It follows that $\langle \mathcal{F}', V' \rangle$ is preferred over \mathcal{M} . Now since \mathcal{M} is a minimal model of $\tau(X)$, $\langle \mathcal{F}', V' \rangle \not\models \tau(X)$. But applying the skeleton Lemma we have for any φ ,

$$\langle \mathcal{F}', V' \rangle \models \tau(\varphi) \text{ iff } \rho \langle \mathcal{F}', V' \rangle \models \varphi \text{ iff } \mathcal{N} \models \varphi.$$

This shows that $\mathcal{N} \not\models X$ contradicting our assumption.

The ‘only if’ direction is similar and left to the reader. \square

Theorem 3. τ^+ is an embedding of equilibrium logic into non-monotonic $wK4f$. In particular we have for any set X of HT-formulas, and formula φ

$$X \vdash \varphi \text{ iff } \tau(X) \vdash_{S4F^*} \tau(\varphi) \text{ iff } \tau^+(X) \vdash_{wK4f^*} \tau^+(\varphi).$$

Proof. Apply Lemmas 3 and 4 together with Theorem 2. \square

To complete our picture of modal embeddings, let us turn to two well-known strengthenings of $S4F$ and $wK4f$ respectively. $SW5$ is the extension of $S4F$ whose frames consist of a single reflexive point that sees a cluster (in Fig. 1, W_1 becomes a single point). Likewise $KD45$ is captured by frames that comprise a single irreflexive point that sees a cluster (in Fig. 2, W_1 becomes an irreflexive point while W_2 now contains only reflexive points). The non-monotonic version of $KD45$ is well-known as being (equivalent to) autoepistemic logic. Since the frames for $SW5$ are similar, but reflexive, its non-monotonic version, $SW5^*$ has been called *reflexive* autoepistemic logic, [14].

It is easy to see that our previous embeddings also hold for $SW5$ and $KD45$ together with their autoepistemic extensions (notice that these logics fall under the scope of Schwarzs theorem and so they can be characterised in terms of minimal models). The proofs are entirely analogous. Without stating these properties as theorems, we merely present them as the picture in Fig. 3.

$$\begin{array}{ccccc} EL & \xrightarrow{\tau} & SW5^* & \xrightarrow{+} & KD45^* \\ \uparrow & & \uparrow & & \uparrow \\ HT & \xrightarrow{\tau} & SW5 & \xrightarrow{+} & KD45 \end{array}$$

Fig. 3.

The close relation between autoepistemic and reflexive autoepistemic logic was already studied in [?].

5 Deriving embeddings for logic programs

Our next task is to examine some of the familiar embeddings of logic programs under stable model and answer set semantics to nonmonotonic modal logics. We consider

some of the most general ones here and show how to derive them from our main theorem.

Even before the birth of the stable model semantics for logic programs in [21] efforts were made to connect negation as failure with modal logic. Early steps were taken by Gabbay [24] and Gelfond [20].⁸ Gabbay used the provability logic GL (and a specially adapted extension) to interpret negation as failure, while Gelfond provided a connection between SLDNF-resolution for stratified programs and provability in autoepistemic logic, a similar connection being maintained for arbitrary normal programs under the stable model semantics developed with Lifschitz around the same time [21].

The concept of stable model or answer set was soon extended to embrace more general kinds of logic programs, containing disjunctive rules and a second, strong negation operator, though it was not immediately apparent how these extensions could be related to non-monotonic modal systems [22]. Answers were provided at the 2nd International LPNMR workshop held in Lisbon in June 1993. No fewer than three papers by five authors proposed similar embeddings of answer set semantics into autoepistemic logic [14, 13, 3], while two of these papers also dealt with the relation to non-monotonic SW5, alias “reflexive” autoepistemic logic [14, 13].

5.1 Answer sets and SW5 expansions

The main results of [14, 13] concern disjunctive logic programs. They consist of rules α , that, if written as logical formulas, have the shape

$$p_1 \wedge \dots \wedge p_m \wedge \neg p_{m+1} \wedge \dots \wedge \neg p_n \rightarrow q_1 \vee \dots \vee q_k \quad (11)$$

where the p_i and q_j are atoms.⁹ To establish modal embeddings they consider the following translation of a rule α of form (11), which we denote by $\sigma(\alpha)$:

$$Lp_1 \wedge \dots \wedge Lp_m \wedge L\neg p_{m+1} \wedge \dots \wedge L\neg p_n \rightarrow Lq_1 \vee \dots \vee Lq_k \quad (12)$$

A disjunctive logic program is a set of such rules and if Π is a disjunctive program let $\sigma(\Pi) := \{\sigma(\alpha) : \alpha \in \Pi\}$. The results of [13] and [14] are quite similar but they are presented and proved somewhat differently. [13] uses the bi-modal system of minimal belief and negation-as-failure (as does [3]) while in [14] Marek and Truszczyński use the method of preferred models. We state the main property using their formulation.

Proposition 1 ([14, 13]). *Let Π be a logic program and T a set of atoms. Then T is an answer set for Π if and only if $ST(T)$ is an SW5-expansion of $\sigma(\Pi)$.*

If X is a set of non-modal formulas, $ST(X)$ stands for the unique stable theory E whose non-modal part is precisely $Cn(X)$, the classical propositional consequences of X .

It is well-known that equilibrium logic generalises stable model semantics to the full language of propositional logic. In particular, for any kind of logic program Π , a

⁸ Gabbay’s paper from 1991 was circulated in draft form from the mid-80s.

⁹ They also include the case where the p_i, q_j can be atoms or their strong negations but let us postpone the topic of strong negation to a later section.

set T of atoms is a stable model of Π if and only if there is an equilibrium model \mathcal{M} of Π (writing rules of Π as logical formulas) such that for any atom p , $p \in T$ iff $\mathcal{M} \models p$. If we represent HT -models as pairs $\langle H, T \rangle$ where H is the set of atoms verified ‘here’ and T is the set of atoms verified ‘there’, then T is an answer set of Π iff $\langle T, T \rangle$ is an equilibrium model of Π . It follows that all embeddings of equilibrium logic include as a special case corresponding embeddings of logic programs under stable models.

To derive Proposition 1, it suffices to note that equilibrium models and answer sets coincide and to show that the embedding σ is closely related to τ .

Proposition 2. *For any program rule α of form (11), in $SW5$, $\tau(\alpha) \equiv L\sigma(\alpha)$.*

Proof. Let α be of form (11). Then by inspection $\tau(\alpha)$ is the formula

$$L(Lp_1 \wedge \dots \wedge Lp_m \wedge L\neg Lp_{m+1} \wedge \dots \wedge L\neg Lp_n \rightarrow Lq_1 \vee \dots \vee Lq_k) \quad (13)$$

which is exactly $L\sigma(\alpha)$. \square

Since answer sets correspond to equilibrium models, the embedding depicted in Fig. 3 establishes the relation between answer sets of a disjunctive program Π and the reflexive autoepistemic expansions of $L\sigma(\Pi)$, where for any set of formulas Γ , we put $L\Gamma = \{L\varphi : \varphi \in \Gamma\}$. Although in extensions of $S4$, $\tau(\Pi)$ and $\sigma(\Pi)$ are not generally equivalent, for non-monotonic $SW5$ there is no difference. In particular it is easy to see that these sets of formulas have the same $SW5$ -expansions. Recall that $SW5$ -expansions are also called reflexive autoepistemic expansions because they satisfy the following condition [13, 14]: E is an $SW5$ -expansion of I if and only if

$$E = Cn(I \cup \{\varphi \leftrightarrow L\varphi : \varphi \in E\} \cup \{\neg L\varphi : \varphi \notin E\}) \quad (14)$$

where Cn is ordinary consequence in classical propositional logic.

Lemma 5. *For any set Γ , $L\Gamma$ and Γ have the same $SW5$ -expansions.*

Proof. For any set $T \supseteq I$, it is evident that $Cn(I \cup \{\varphi \leftrightarrow L\varphi : \varphi \in T\})$ and $Cn(LI \cup \{\varphi \leftrightarrow L\varphi : \varphi \in T\})$ are the same.

To derive Proposition 1 one may use the following simple lemma.

Lemma 6. *Let \mathcal{M} be a minimal $SW5$ -model and φ a modal-free formula. Then $\mathcal{M} \models \varphi$ if and only if $\mathcal{M} \models \tau(\varphi)$.*

This can be shown by induction on φ . Now notice that if T is an answer set of Π and hence $\langle T, T \rangle$ is an equilibrium model, then the formulas true in $\langle T, T \rangle$ are precisely the classical consequences $Cn(T)$ of T . Now we apply Lemmas 4 and 6. Then \mathcal{M} is a minimal model of $\tau(\Pi)$ if and only if $\rho(\mathcal{M})$ is an equilibrium model of Π , and $\rho(\mathcal{M}) \models \varphi$ iff $\mathcal{M} \models \tau(\varphi)$ iff $\mathcal{M} \models \varphi$, for modal-free φ . Since the stable expansion is given by the formulas true in the minimal model, we have established that the modal-free formulas true in the stable expansion are precisely those in $Cn(T)$, where $\langle T, T \rangle$ is $\rho(\mathcal{M})$ and T is an answer set of Π . So T is answer set for Π if and only if $ST(T)$ is an $SW5$ -expansion of $\tau(\Pi)$. Proposition 1 then follows by Lemma 5 and Proposition 2.

5.2 Autoepistemic logic

There have been several embeddings of stable model semantics into autoepistemic logic. For the case of disjunctive logic programs with rules of shape (11), the following translation Tr was used by [13, 14]: for α of the form (11), $Tr(\alpha)$ is the expression:

$$(p_1 \wedge Lp_1) \wedge \dots \wedge (p_m \wedge Lp_m) \wedge \neg Lp_{m+1} \wedge \dots \wedge \neg Lp_n \rightarrow (q_1 \wedge Lq_1) \vee \dots \vee (q_k \wedge Lq_k) \quad (15)$$

The result obtained by [13, 14] can be stated thus:

Theorem 4 ([13, 14]). *Let Π be a logic program. A set of atoms T is a stable model of Π if and only if $ST(T)$ is an autoepistemic expansion of $Tr(\Pi)$.*

Proof. It suffices to show that $Tr(\alpha)$ is equivalent to $\tau^+(\alpha)$. From the previous case it is clear that we can simplify $\tau(\alpha)$ to the expression (13). Applying the $^+$ translation to this we obtain the following form for $\tau^+(\alpha)$:

$$(p_1 \wedge Lp_1) \wedge \dots \wedge (p_m \wedge Lp_m) \wedge (\neg(p_{m+1} \wedge Lp_{m+1}) \wedge L(\neg(p_{m+1} \wedge Lp_{m+1}))) \\ \wedge \dots \wedge (\neg(p_n \wedge Lp_n) \wedge L(\neg(p_n \wedge Lp_n))) \rightarrow (q_1 \wedge Lq_1) \vee \dots \vee (q_k \wedge Lq_k)$$

This can be simplified, in particular by noting that in $KD45$ the equivalence

$$\neg L\varphi \equiv L(\neg(\varphi \wedge L\varphi))$$

holds. Substituting for the RHS of this equivalence, the middle terms of $\tau^+(\alpha)$ become

$$\neg(p_{m+1} \wedge Lp_{m+1}) \wedge \neg Lp_{m+1}$$

etc, which by propositional logic is equivalent to $\neg Lp_{m+1}$. So $\tau^+(\alpha)$ becomes

$$(p_1 \wedge Lp_1) \wedge \dots \wedge (p_m \wedge Lp_m) \wedge \neg Lp_{m+1} \wedge \dots \wedge \neg Lp_n \rightarrow (q_1 \wedge Lq_1) \vee \dots \vee (q_k \wedge Lq_k) \quad (16)$$

as required.

6 Draft end here - to be completed

In the mid-1980s efforts were made to understand negation-as-failure in logic programming in terms of provability in modal logics (Gabbay, ca. 1985, Gelfond, 1987). The second of these works was based on the non-monotonic system of autoepistemic logic and led a short time later to the discovery of stable model semantics that subsequently formed the basis for answer set programming. With stable model semantics, the new area of logic programming and non-monotonic reasoning quickly emerged, its first international workshop being held in 1991. A feature of the emergent area was the study of the mathematical relations between different non-monotonic formalisms, especially between the semantics of stable models and various non-monotonic modal logics. Given the embeddings provided by Schwarz, Gelfond, Lifschitz, Marek and Truszczyński, Chen and others into systems of modal logic, the modal interpretation of logic programming rules became the dominant view for most of the 1990s.

The translation used by Goedel in 1933 to embed Heyting’s intuitionistic logic into a modal provability logic equivalent to Lewis’s S4. The splitting translation, known since the mid-1970s, allows one to embed extensions of S4 into extensions of the non-reflexive logic, K4. A famous early result of this kind is the embedding by Kuznetsov and Muravitsky (1976) of the Grzegorczyk logic Grz into the Goedel-Loeb provability logic GL. By composing the two translations one can obtain (Goldblatt, 1978) an adequate provability interpretation of Heyting’s calculus within GL, the logic that Solovay (1976) showed to capture precisely the provability predicate of Peano arithmetic.

In this paper we explore an approach to minimal belief that borrows the basic ideas of minimal knowledge studied by Schwarz and Truszczyński [18, 19]. At the same time we show how the logics underlying minimal knowledge and belief are related to minimal topologies using the well-known methods for obtaining logics from topologies described by [34] and [7]. We start with a brief review of these basic ideas.¹⁰

Minimal Knowledge and Minimal Belief The paradigm of minimal knowledge derives from the well-known work of Halpern and Moses, especially [11], later extended and modified in works such as [30, 27, 28] and others. Many approaches are based on Kripke-*S5*-models with a universal accessibility relation and the minimisation of knowledge is represented by maximising the set of possible worlds with respect to inclusion. In general, this has the effect of minimising objective knowledge, ie knowledge of basic facts and propositions. A somewhat different approach was developed by Schwarz and Truszczyński [18] and can be seen as a special case of the very general method of Shoham [30] for obtaining different concepts of minimality by changing the sets of models and preference relations between them. The initial models considered by [18] (see also [39, 19]) consist of not one (*S5*) cluster but rather two clusters arranged in such a way that all worlds in one cluster are accessible from all worlds in the other (but not vice versa). In Figure 1 clusters are labelled W_1, W_2 , all points are reflexive and

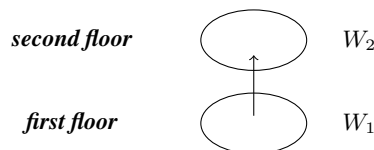


Fig. 4.

every point in W_2 is accessible from every point in W_1 . We call W_1, W_2 respectively the *first* and the *second floor* of the model. The former may be empty but the latter not.

¹⁰ The authors are grateful to anonymous reviewers whose comments helped to improve the readability of the paper. The second author is grateful to Leo Esakia for discussions on monotonic $wK4f$ and its connections with topology and also to David Gabelaia due to whom the axiom f is much more readable. This research has been partially supported by the MCICINN projects TIN2006-15455, TIN2009-14562-CO5, and CSD2007-00022.

In [18, 19], given a background theory or knowledge set I , minimal knowledge (with respect to I) is captured by an $S5$ -model of the theory, say M , but now the idea of minimality is that there should be no two-floor model M' as in Figure 1 of the same theory I , where M coincides with the restriction of M' to the second floor W_2 , and W_1 is smaller in the sense that it fails to verify some objective (non-modal) sentence true in M . Schwarz and Truszczyński argue that this approach to minimal knowledge has some important advantages over the method of [11] and they study its properties in depth, in particular showing that while the two-floor models correspond to the modal logic $S4F$ first studied by Segerberg [15], minimal knowledge is precisely captured by non-monotonic $S4F$. In [19] they show that non-monotonic $S4F$ captures, under some intuitive encodings, several important approaches to knowledge representation. They include disjunctive logic programming under answer set semantics [?], (disjunctive) default logic [23], [25], the logic of grounded knowledge [27], the logic of minimal belief and negation as failure [28] and the logic of minimal knowledge and belief [19]. Recently, Truszczyński [31] and Cabalar [32] have revived the study of $S4F$ in the context of a general approach to default reasoning.

Logics via Topology Alfred Tarski [16], together with Chen McKinsey [34, 35], laid the foundations for the algebraic and topological study of intuitionistic and modal logics. The basic idea, recalled and developed in a recent paper by Leo Esakia [7], is that from an arbitrary topological space X we can generate three different algebraic structures each giving rise to different logical systems.¹¹ By considering the algebra of open sets, $Op(X)$, one is led to the well-known Heyting algebra that forms a semantical basis for intuitionistic logic. By considering the closure algebra, $(\mathcal{P}(X), c)$ one is led to the modal system $S4$.¹²

The third path from topology to logic is via what are known as *derivative algebras*, $(\mathcal{P}(X), der)$. These are Boolean algebras with a unary operation der representing topological derivation: if A is a subset of X then $der(A)$ is the set of all accumulation or limit points of A . The derivative algebra $(\mathcal{P}(X), der)$ gives rise to the modal logic $wK4$, a slightly weaker version of the logic $K4$, that was first studied from a topological point of view in [4] (see [7] for a detailed overview).

All three paths to logic are of interest for the modelling of agents' reasoning, their knowledge and beliefs in AI. Intuitionistic logic and its extensions capture different forms of constructive reasoning, while extensions of $S4$, including $S5$, have formed the basis for epistemic logics of knowledge. On the other hand, extensions of $wK4$ may be considered good candidates for doxastic logics of belief inasmuch as the axiom $\Box p \rightarrow p$ does not hold. In fact, the standard doxastic logic $KD45$ is one such extension of $wK4$.

From the viewpoint of non-monotonic reasoning, there is a special interest in examining the logics that arise as above from topological spaces X that are *minimal*, that is where X has only three open sets. In the first case, we obtain the three-element Heyting

¹¹ For the basic notions of topology see eg [12] or any appropriate textbook.

¹² Recall that a Heyting algebra $(\mathbf{H}, \vee, \wedge, \rightarrow, \perp)$ is a distributive lattice with smallest element \perp containing a binary operation \rightarrow such that $x \leq a \rightarrow b$ iff $a \wedge x \leq b$. $(\mathbf{B}, \vee, \wedge, -, c)$ is a closure algebra if $(\mathbf{B}, \vee, \wedge, -)$ is a Boolean algebra and c is a closure operator satisfying: $a \leq ca$, $cca = ca$, $c(a \vee b) = ca \vee cb$, $c\perp = \perp$.

algebra that captures a logic known as *here-and-there*, HT , the maximal intermediate logic that is properly contained in classical logic. The well-known non-monotonic extension of HT called *equilibrium logic* [33] provides a logical foundation for reasoning with the stable model semantics of logic programs and thus for the popular approach to knowledge representation and declarative problem solving known as *answer set programming*, ASP. Starting from a minimal topological space and using instead the idea of closure algebras one arrives at $S4F$, a reflexive normal modal logic first studied by Segerberg [15]. We have already observed how non-monotonic $S4F$ relates to minimal knowledge and is important in knowledge representation and reasoning.

In this paper we study the third path from topology to logic based on minimal topological spaces. This yields a logic that we call $wK4f$. Our main motivation is that this logic (and some close variants) can serve to model minimal belief, in the same way as $S4F$ captures minimal knowledge. We would like to emphasise again that these ideas of minimal knowledge and belief are based on properties of models (and what they verify) and not on the shape of modal axioms. While obtaining a complete axiomatisation is therefore an important and even essential part of our study, it is not the axioms themselves that motivate the choice of logic. They provide a compact formulation of a calculus rather than a direct formalisation of some or other intuitive property of belief. The connections with minimal belief will be explored in the second half of the paper, in Section 10, while the first parts of the paper are devoted to the study of $wK4f$ itself.

As we have seen, $S4F$ is captured by Kripke frames consisting of two clusters connected by an accessibility relation. In the case of $wK4f$ the picture is similar except

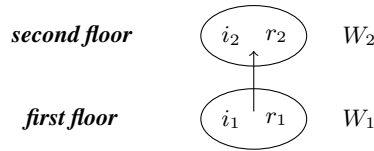


Fig. 5.

that we drop the condition of reflexivity on frames: in Fig. 2 some points in W_1, W_2 may now be irreflexive (where i and r label this difference). Since $S4F$ and $wK4f$ are closely related, many results can be transferred from one to the other.

The paper is organised in the following way. In section 2 we present the syntax and Kripke semantics of $wK4f$. We prove completeness and the finite model property. In section 3 we characterise finite one-step, weakly-transitive frames and their bounded morphisms in terms of quadruples of natural numbers. In section 4 we prove a main theorem of the paper, which states that $wK4f$ is the (sound and complete) logic of all minimal topological spaces. Section 5 describes non-monotonic $wK4f$ and relates it to the idea of minimal belief. In the last section we state some conclusions and mention topics for future work.

7 The modal logic $wK4f$

Following the Tarski/McKinsey suggestion to treat modality as the derivative of the topological space [34], Esakia in [4, 37] introduced $wK4$ as the modal logic of all topological spaces, with the desired (derivative operator) interpretation of the modal \Diamond . $wK4f$ is a normal modal logic obtained by adding the axiom weak- F to the modal logic $wK4$. $wK4f$ is a weaker logic than $S4F$ discussed in Segerberg [15] since it doesn't satisfy the axiom T . However since the frames of $wK4f$ and $S4F$ are closely related, some results about $wK4f$ may carry over to $S4F$.

Syntax. The normal modal logic $wK4f$ is defined in a basic modal language with an infinite set $Prop$ of propositional letters and connectives \vee, \wedge, \neg, \Box . The axioms are all classical tautologies plus the axioms listed below. Rules of inference are: modus ponens, substitution and necessitation.

$$\begin{aligned} K &: \Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q) \\ w4 &: \Box p \wedge p \rightarrow \Box \Box p \\ f &: p \wedge \Diamond(q \wedge \Box \neg p) \rightarrow \Box(q \vee \Diamond q) \end{aligned}$$

Semantics. Kripke semantics for the modal logic $wK4f$ is provided by frames which have in a weak sense height at most two and which do not allow forking. This is made precise in the following two definitions.

Definition 1 We will say that a relation $R \subseteq W \times W$ is weakly-transitive if $(\forall x, y, z) (xRy \wedge yRz \wedge x \neq z \Rightarrow xRz)$.

Clearly every transitive relation is weakly-transitive as well. Moreover, clusters (where a cluster is a subset of a frame where every two distinct points are related with each other) with weakly transitive relations differ from those with transitive relations in that they allow for irreflexive points. Such clusters will be called *weak-clusters*.

Definition 2 We will say that a relation $R \subseteq W \times W$ is a one-step relation if the following two conditions are satisfied:

- 1) $(\forall x, y, z)((xRy \wedge yRz) \Rightarrow (yRx \vee zRy))$,
- 2) $(\forall x, y, z)((xRy \wedge \neg(yRx) \wedge xRz \wedge y \neq z) \Rightarrow zRy)$.

As the reader can see the first condition restricts the 'strict' height of the frame to two. Where informally by "strict" we mean that the steps are not counted *within* a cluster. The second condition is more complicated. Essentially it restricts the 'strict' width of the frame to one, though more points could be allowed at the bottom.

We briefly recall some standard definitions in modal logic. The pair (W, R) , with W an arbitrary set and $R \subseteq W \times W$ is called a Kripke frame. If we additionally have a third component, a function $V : Prop \times W \rightarrow \{0, 1\}$, then we say that we have a Kripke model $\mathcal{M} = (W, R, V)$.

For a given Kripke model $\mathcal{M} = (W, R, V)$ the satisfaction of a formula at a point $w \in W$ is defined inductively as follows: $\mathcal{M}, w \Vdash p$ iff $V(p, w) = 1$, the Boolean cases

are standard, $\mathcal{M}, w \Vdash \Box\phi$ iff $(\forall v \in W)(wRv \Rightarrow v \Vdash \phi)$. A formula ϕ is valid in a model \mathcal{M} if for every point $w \in W$ we have $\mathcal{M}, w \Vdash \phi$ in this case we write $\mathcal{M} \Vdash \phi$. A formula is valid in a frame if it is valid in every model based on the frame. A formula is valid in a class of frames if it is valid in every frame in the class. Notice that from the definition of \Diamond , ($\Diamond\phi \equiv \neg\Box\neg\phi$) it easily follows that $\mathcal{M}, w \Vdash \Diamond\phi$ iff $\exists v \in W$ such that $wRv \wedge v \Vdash \phi$.

Let \mathcal{K} denote the class of all one-step and weakly-transitive Kripke frames. The following theorem links the logic $wK4f$ with the class \mathcal{K} . The proof uses standard modal logic completeness techniques, so we will not enter into all the details.

Theorem 3 *The modal logic $wK4f$ is sound and strongly complete wrt the class \mathcal{K} .*

We give the soundness proof only for the axiom f . For the proofs for other axioms the reader may consult [37].

Proof. Take an arbitrary, weakly-transitive, one-step model (W, R, V) . Assume at some point $w \in W$ it holds that $w \Vdash p \wedge \Diamond(q \wedge \Box\neg p)$. This implies that $w \Vdash p$ and there exists w' such that wRw' , $w' \Vdash q$ and it is not the case that $w'Rw$ (as far as $w' \Vdash \Box\neg p$). Now for an arbitrary v with wRv and $v \neq w'$, by the second condition of definition 2, we have that vRw' , which implies that $v \Vdash \Diamond q$ and hence $w \Vdash \Box(q \vee \Diamond q)$.

For strong completeness assume $I \not\Vdash \phi$. We will construct the one-step and weakly-transitive model $M^c = \{W^c, R^c, V^c\}$ such that $M^c \Vdash I$ and $M^c \not\Vdash \phi$. For M^c we take a standard canonical model ie. $W^c = \{\Gamma \mid \Gamma \vdash I \text{ and } \Gamma \text{ is a maximal consistent set}\}$. The relation is defined in a standard way $\Gamma R^c \Gamma'$ iff $(\forall \alpha)(\Box\alpha \in \Gamma \Rightarrow \alpha \in \Gamma')$ and $V^c(p, \Gamma) = 1$ iff $p \in \Gamma$.

Lemma 4 (Truth Lemma) *For any formula ϕ we have $M^c, \Gamma \Vdash \phi$ iff $\phi \in \Gamma$.*

The proof follows a standard pattern found in modal logic textbooks. As $I \not\Vdash \phi$ we have that $I \cup \{\neg\phi\}$ is consistent, so there exists a maximally consistent set $\Gamma_{\neg\phi}$ containing $I \cup \{\neg\phi\}$ and by the truth lemma this means that $M^c, \Gamma_{\neg\phi} \Vdash \neg\phi$ which completes the proof. The main thing to be checked is that $M^c \in \mathcal{K}$. For weak transitivity of the relation R^c the reader may consult [37]. Let us show that R^c is a one-step relation.

First let us show that R^c satisfies the first condition of definition 2. For the contradiction assume there exist three distinct points $\Gamma, \Gamma', \Gamma'' \in W^c$ such that $\Gamma R^c \Gamma' \wedge \Gamma' R^c \Gamma''$ and $\neg(\Gamma' R^c \Gamma) \wedge \neg(\Gamma'' R^c \Gamma')$. This means that there is a formula ψ such that $\Box\psi \in \Gamma'$ and $\neg\psi \in \Gamma$ and there is a formula ϕ such that $\Box\phi \in \Gamma''$ and $\neg\phi \in \Gamma'$ and as $\Gamma' \neq \Gamma''$ there exists a formula γ with $\gamma \in \Gamma'$ and $\neg\gamma \in \Gamma''$. From these assumptions we have that $(\neg\phi \wedge \gamma) \wedge \Box\neg\neg\psi \in \Gamma'$. Now as $\Gamma R^c \Gamma'$ we have that $\Diamond((\neg\phi \wedge \gamma) \wedge \Box\neg\neg\psi) \in \Gamma$ and as $\neg\psi \in \Gamma$ we have that $\Diamond((\neg\phi \wedge \gamma) \wedge \Box\neg\neg\psi) \wedge \neg\psi \in \Gamma$. Applying axiom f (with $p = \neg\psi, q = \neg\phi \wedge \gamma$) we get that $\Box((\neg\phi \wedge \gamma) \vee \Diamond(\neg\phi \wedge \gamma)) \in \Gamma$. Hence as $\Gamma R^c \Gamma''$ (because of weak transitivity) we have that $(\neg\phi \wedge \gamma) \vee \Diamond(\neg\phi \wedge \gamma) \in \Gamma''$. On the other hand $\Diamond(\neg\phi \wedge \gamma) \notin \Gamma''$ since $\Box\phi \in \Gamma''$ and $\neg\phi \wedge \gamma \notin \Gamma''$ because $\neg\gamma \in \Gamma''$. Hence we get a contradiction.

Now let us show that R^c satisfies the second condition of definition 2. Again the proof is by contradiction. Assume there exist three distinct points $\Gamma, \Gamma', \Gamma'' \in W^c$ such that $\Gamma R^c \Gamma' \wedge \Gamma R^c \Gamma'' \wedge \neg(\Gamma' R^c \Gamma)$ and $\neg(\Gamma'' R^c \Gamma')$. This means that there is a formula

ψ such that $\Box\psi \in \Gamma'$ and $\neg\psi \in \Gamma$ and there is a formula ϕ such that $\Box\phi \in \Gamma''$ and $\neg\phi \in \Gamma'$ and as $\Gamma' \neq \Gamma''$ there exists a formula γ with $\gamma \in \Gamma'$ and $\neg\gamma \in \Gamma''$. From these assumptions we have that $(\neg\phi \wedge \gamma) \wedge \Box\neg\psi \in \Gamma'$. Now as $\Gamma R^c \Gamma'$ we have that $\Diamond((\neg\phi \wedge \gamma) \wedge \Box\neg\psi) \in \Gamma$ and as $\neg\psi \in \Gamma$ we have that $\Diamond((\neg\phi \wedge \gamma) \wedge \Box\neg\psi) \wedge \neg\psi \in \Gamma$. Applying axiom f we get that $\Box((\neg\phi \wedge \gamma) \vee \Diamond(\neg\phi \wedge \gamma)) \in \Gamma$. Hence as $\Gamma R^c \Gamma''$ we have that $(\neg\phi \wedge \gamma) \vee \Diamond(\neg\phi \wedge \gamma) \in \Gamma''$. On the other hand $\Diamond(\neg\phi \wedge \gamma) \notin \Gamma''$ since $\Box\phi \in \Gamma''$. Nor can we have $\neg\phi \wedge \gamma \in \Gamma''$ because $\neg\gamma \in \Gamma''$. Hence we get a contradiction.

Theorem 5 *The modal logic $wK4f$ is sound and complete wrt the class of all **finite** one-step and weakly-transitive Kripke frames.*

8 Finite, rooted, weakly-transitive and one-step Kripke frames.

We saw from (Theorem 5), which we do not prove in this paper, that the class of finite, weakly-transitive and one-step Kripke frames fully captures the modal logic $wK4f$. From general theorems in modal logic it is well known that this class can be reduced to a smaller class of frames which are rooted so that the completeness theorem still holds. In this section we characterise finite, rooted, weakly-transitive and one-step Kripke frames in terms of quadruples of natural numbers.

Definition 6 *The upper cone of a set $A \subseteq W$ in a weakly-transitive Kripke frame (W, R) is defined as a set $R(A) = \bigcup\{y : x \in A \& xRy\} \cup A$.*

Observe that the general definition of upper cone in an arbitrary Kripke frame is given in terms of the reflexive, transitive closure of a relation, while Definition 6 is a simplified version for the particular case of weakly transitive frames.

Definition 7 *A Kripke frame (W, R) is called rooted if there exists a point $w \in W$ such that the upper cone $R(\{w\}) = W$; w is called the root of the frame.*

Let N^4 be the set of all quadruples of natural numbers and let $\mathcal{N}^4 = N^4 - \{(n, m, 0, 0) | n, m \in N\}$. The following theorem states that the set \mathcal{K}_r of all finite, rooted, one-step, weakly-transitive frames considered up to isomorphism can be seen as the set \mathcal{N}^4 .

Theorem 8 *There is a one-to-one correspondence between the set \mathcal{K}_r and the set \mathcal{N}^4 .*

Proof. We know that any one-step frame has "strict" width one and "strict" height less than or equal to two (We didn't give the formal definition of "strict" height and width, but it should be clear from the intuitive explanation after the definitions 2 and 1 what we mean by this). If additionally we have that the frame is rooted, the case where strict width is greater than one at the bottom is also restricted. It is not difficult to verify that any such frame (W, R) is of the form (W_1, W_2) , where $W_1 \cup W_2 = W$, $W_1 \cap W_2 = \emptyset$ and $(\forall u \in W_1, \forall v \in W_2)(uRv)$. Besides because of the weak-transitivity, we have that $(\forall u, u' \in W_1)(u \neq u' \Rightarrow uRu')$ and the same for every two points $v, v' \in W_2$. Pictorially any rooted, weak-transitive and one-step Kripke frame can be represented as in Figure 2. Again we call W_1 the **first floor** and W_2 the **second floor** of the frame

(W, R) . Notice that W_1 or W_2 may be equal to \emptyset ie the frame has only one floor. In this case we treat the only floor of the frame as the second floor.

Now let us describe how to construct the function from \mathcal{K}_r to \mathcal{N}^4 . With every frame $(W, R) \in \mathcal{N}^4$ we associate the quadruple (i_1, r_1, i_2, r_2) , where i_1 is the number of irreflexive points in W_1 , r_1 is the number of reflexive points in W_1 , i_2 is the number of irreflexive points in W_2 and r_2 is the number of reflexive points in W_2 . We will call the quadruple (i_1, r_1, i_2, r_2) the **characteriser** of the frame (W, R) . In case the frame (W, R) has only one floor, by our earlier remark it is treated as the frame (\emptyset, W) . Hence its characteriser has the form $(0, 0, i, r)$. Now it is clear that the correspondence described above defines a function from the set \mathcal{K}_r to the set \mathcal{N}^4 . We denote this function by Ch .

Claim 1: Ch is injective. Take any two distinct finite, rooted, weakly-transitive, one-step Kripke frames (W, R) and (W', R') . That they are distinct in \mathcal{K}_r means that they are non-isomorphic ie either $|W| \neq |W'|$ or $R \not\cong R'$. In the first case it is immediate that $Ch(W, R) \neq Ch(W', R')$ since $|W| = i_1 + i_2 + r_1 + r_2$. In the second case we have three subcases:

1) $|W_1| \neq |W'_1|$. In this subcase $i_1 + r_1 \neq i'_1 + r'_1$ and hence $Ch(W, R) \neq Ch(W', R')$.

2) The number of reflexive (irreflexive) points in $|W_1|$ differs from the number of reflexive (irreflexive) points in $|W'_1|$. In this subcase $i_1 \neq i'_1$ and again $Ch(W, R) \neq Ch(W', R')$.

3) The number of reflexive (irreflexive) points in $|W_2|$ differs from the number of reflexive (irreflexive) points in $|W'_2|$. This case is analogous to the previous one.

It is straightforward to see that if none of these cases above occur ie $|W| = |W'|$, $|W_1| = |W'_1|$, $|\{w|w \in W_1 \wedge wRw\}| = |\{w'|w' \in W'_1 \wedge w'R'w'\}|$ and $|\{w|w \in W_2 \wedge wRw\}| = |\{w'|w' \in W'_2 \wedge w'R'w'\}|$ then (W, R) is isomorphic to (W', R') and hence $(W, R) = (W', R')$ in \mathcal{K}_r .

Claim 2: Ch is surjective. Take any quadruple $(i_1, r_1, i_2, r_2) \in \mathcal{N}^4$. Let us show that the pre-image $Ch^{-1}((i_1, r_1, i_2, r_2))$ is not empty. Take the frame $(W, R) = (W_1, W_2)$, where $|W_1| = i_1 + r_1$, $|W_2| = i_2 + r_2$, W_1 contains i_1 irreflexive and r_1 reflexive points and $|W_2|$ contains i_2 irreflexive and r_2 reflexive points. Then by the definition of Ch , we have that $Ch(W, R) = (i_1, r_1, i_2, r_2)$.

9 Connection with minimal topological spaces

In this section we show that $wK4f$ is the modal logic of minimal topological spaces. A topological space is minimal if it has only three open sets. It is well known that there is a bijection between Alexandrof spaces and weakly-transitive, irreflexive Kripke frames and this bijection preserves modal formulas. In this section we show that the special case of this correspondence for minimal topological spaces gives one-step, irreflexive and weakly-transitive relations as a counterpart. As a corollary it follows that the logic $wK4f$ is sound and complete wrt the class of minimal topological spaces.

Theorem 9 *There is a one-to-one correspondence between the class of all **irreflexive**, weakly-transitive, finite, rooted, one-step Kripke frames and the class of all finite minimal topological spaces.*

Proof. Assume (W, R) is a finite, rooted, irreflexive, weakly-transitive and one-step relational structure. (Note that as the frame is irreflexive its characteriser has the form $(i_1, 0, i_2, 0)$, where $i_1 + i_2 = |W|$.) Let W_1 be the first floor and W_2 the second floor of the frame, then the topology we construct is $\{W, \emptyset, W_2\}$. It is immediate that the space (W, Ω_R) , where $\Omega_R = \{W, \emptyset, W_2\}$, is a minimal topological space.

Let us show that the correspondence we described is injective. Take two arbitrary distinct irreflexive, finite, rooted, weakly-transitive frames (W, R) and (W', R') . As they are distinct, either $W \neq W'$ or $R \neq R'$. In the first case it is immediate that $(W, \Omega_R) \neq (W', \Omega_{R'})$. In the second case as both R and R' are irreflexive the second floors are not the same, so $W_2 \neq W'_2$ and hence $\Omega_R \neq \Omega_{R'}$.

For surjectivity take an arbitrary minimal topological space (W, Ω) , where $\Omega = \{W, \emptyset, W_0\}$ for some subset $W_0 \subseteq W$. Take the frame (W, R) , where $R = (W_0 \times W_0 - \{(w, w) | w \in W_0\}) \cup (-W_0 \times -W_0 - \{(w, w) | w \in -W_0\}) \cup \{(w, w') | w \in -W_0, w' \in W_0\}$. In words every two distinct points are related in W_0 by R and the same in the complement $-W_0 = W - W_0$, besides every point from the $-W_0$ is related to every point from W_0 . What we get is the rooted one-step relation which is weakly-transitive, with the second floor equal to W_0 . As we didn't allow wRw for any point $w \in W$, the relation R is also irreflexive.

We now give the definition of a derived set (or set of accumulation points) of a set in a topological space. This definition is needed to give the derived set semantics of modal formulas in an arbitrary topological space.

Definition 10 *Given a topological space (W, Ω) and a set $A \subseteq W$ we will say that $w \in W$ is an accumulation point of A if for every neighborhood U_w of w the following holds: $U_w \cap A - \{w\} \neq \emptyset$. The set of all accumulation points of A will be denoted by $der(A)$ and will be called the derived set of A .*

Below we give the definition of satisfaction of modal formulas.

Definition 11 *A topological model (W, Ω, V) is a triple, where (W, Ω) is a topological space and $V : Prop \rightarrow P(W)$ is a valuation function. Satisfaction of a modal formula in a topological model (W, Ω, V) at a point $w \in W$ is defined by:*

$$w \Vdash p \text{ iff } w \in V(p) ; w \Vdash \Diamond p \text{ iff } w \in der(V(p)),$$

Boolean cases are standard. Validity of a formula in a topological space and class of topological spaces is defined in a standard way

Fact 12 *Let (W, R) be a finite, weakly-transitive and irreflexive frame and let (W, Ω_R) be its Alexandrof space. For every modal formula α the following holds:*

$$(W, R) \Vdash \alpha \text{ iff } (W, \Omega_R) \Vdash \alpha.$$

Note that here \Vdash on the left hand side denotes the validity in Kripke frames while on the right hand side it denotes the validity in topological frames in derived set semantics.

Theorem 13 *The modal logic $wK4f$ is sound and complete with respect to the class of all minimal topological spaces.*

Proof. Soundness can be checked directly so we do not prove it here. For completeness assume $\not\models \phi$. By theorem 5 there exists a finite, one-step, weakly-transitive frame (W, R) which falsifies ϕ . Assume that $Ch(W, R) = (i_1, r_1, i_2, r_2)$. It is not difficult to check that (W, R) is a p -morphic image of (W', R') , where $(W', R') = Ch^{-1}(i_1 + 2 \times r_1, 0, i_2 + 2 \times r_2, 0)$. Roughly speaking the main idea here is that two distinct irreflexive points from first floor (second floor) of (W', R') are mapped to one reflexive point of first floor (second floor) of (W, R) . So each reflexive point in (W, R) has two irreflexive preimages and each irreflexive point in (W, R) has one irreflexive point as a preimage. Now as you can see on each floor in (W', R') there are enough irreflexive points to cover both reflexive and irreflexive point of the corresponding floor in (W, R) . So we have a surjection. To check that the described function satisfies back and forth conditions of the p -morphism is left to the reader. The surjection implies that $(W', R') \not\models \phi$. Now as far as (W', R') is irreflexive, the result immediately follows from theorem 9, and the fact 12.

10 Minimal Belief and Non-Monotonic $wK4f$

It is often held that $KD45$ represents an adequate logic for belief. One motivation for this is that it allows positive and negative introspection and additionally $\Box p \rightarrow p$ is not derivable in the logic. A Kripke model \mathcal{M} for $KD45$ consists of cluster W plus one irreflexive point w so that w is related to every point in W but no point in W is related to w . In other words the first floor of \mathcal{M} is one irreflexive point and the second floor is a cluster. The belief set of an agent is obtained as a theory of the second floor. Indeed $\varphi \in Th(W)$ iff $\mathcal{M} \models \Box \varphi$.¹³ In particular minimisation is relative to some base set I of beliefs. The aim is to capture a set which contains I , is closed under positive and negative introspection, is closed under logical deduction and does not contain anything superfluous. One way to obtain such a set is to consider a $KD45$ Kripke model \mathcal{M} such that $v \models I$ for every $v \in W$ and extend the cluster W by adding points which still make I true. As a result the set of objective facts true in every world will be reduced while the starting beliefs I will stay unchanged. This approach is applied in [11] to knowledge sets, but as discussed in [19] it has some unintuitive consequences. For this reason we follow the pattern of [19] which relies on the idea that an agent's belief is dependent not only on the objective facts but also on the things that are believed by agent. More concretely we minimise the belief set by adding worlds on the first floor of the model \mathcal{M} leaving the second floor untouched. This form of minimal model semantics provides an alternative way of minimising belief and I -expansions for $wK4f$ are exactly minimal belief sets. This is the chief motivation for considering non-monotonic $wK4f$ to be a good candidate for the logic of minimal belief.

Formally we want to relate non-monotonic $wK4f$ to the idea of *minimal model* introduced and characterised in [39]. However we cannot directly apply the general result (Theorem 3.1) of [39] since that theorem refers to what are called *cluster-closed* logics. Instead we can adapt Schwarz's techniques to our case, starting with the definition of preferred model for a class \mathcal{K} . The preference relation is between one-floor $S5$ -models

¹³ Recall that the idea of minimisation applies to the belief set of an agent.

and two-floor models and only two-floor models can be preferred over $S5$ -models. For example we can not compare two one-floor models with each other.

Definition 14 We say that a two-floor model $\mathcal{N} = (N, S, U)$ is preferred over $S5$ -model $\mathcal{M} = (W, R, V)$ if:

- a) There is a propositional formula ψ such that $\mathcal{M} \models \psi$ and $\mathcal{N} \not\models \psi$,
- b) (W, R) is the second floor of (N, S) and V equals to the restriction of U to the second floor. Briefly, \mathcal{M} is the model which is obtained by deleting the first floor in \mathcal{N} .

We next define the notion of minimal model that is central for the semantics of non-monotonic modal logics.

Definition 15 An $S5$ -model $\mathcal{M} = (W, R, V)$ is called a \mathcal{K} -minimal model for the set of formulas I if $\mathcal{M} \models I$ and for every preferred model $\mathcal{N} \in \mathcal{K}$ we have $\mathcal{N} \not\models I$.

Non-monotonic $wK4f$ does not fit the scope of Theorem 3.1 [39] because the class \mathcal{K} which characterises monotonic $wK4f$ is not cluster closed. In particular some two-floor models in \mathcal{K} may not have a cluster as a maximum. On the other hand every model in \mathcal{K} has a maximal *weak-cluster* (that is a cluster where irreflexive points are allowed or more precisely it is a rooted, symmetric, weakly-transitive frame). For this reason we need to consider weak-cluster closed classes.

Definition 16 Let $\mathcal{N} = (N, S, U)$ be a Kripke model. A nonempty set $W \subseteq N$ is called a final weak-cluster if:

- a) W is an upper cone (def. 6),
- b) W is weak-cluster,
- c) For every $v \in N - W$ and for every $w \in W$, vRw .

It is immediate from Definition 16 and from Theorem 8 that every rooted, weakly-transitive, one-step frame has a final weak-cluster and it is the second floor (or the only floor) of the frame.

Definition 17 Let $\mathcal{N} = (N, S, U)$ be a Kripke model and let N_2 be its final weak-cluster. Let $\mathcal{M} = (W, R, V)$ be a cluster. By cluster substitution of \mathcal{M} in \mathcal{N} we mean the model $\langle (N - N_2) \cup W, S', V' \rangle$, where for each $w, v \in (N - N_2) \cup W$, $wS'v$ if and only if wSv or $v \in W$ and V' agrees with U on $(N - N_2)$ and agrees with V on W . In other words we substitute the cluster W instead of the weak-cluster N_2 into \mathcal{N} .

Definition 18 By the concatenation of two models (W, R, V) and (N, S, U) with $W \cap N = \emptyset$ we mean the model $(N \cup W, S \cup N \times W \cup R, U \cup V)$.

Definition 19 Let C be a class of models. We say that C is weak-cluster closed if C contains all weak-clusters and for each $\mathcal{N} \in C$, at least one of the following two conditions holds: the concatenation of \mathcal{N} and each cluster belongs to C , or \mathcal{N} has a final weak-cluster and for each $S5$ -model \mathcal{M} , the cluster substitution of \mathcal{M} in \mathcal{N} belongs to C .

It is immediate that \mathcal{K} is weak-cluster closed. As usual, non-monotonic modal logics are defined via the notion of *expansion*.

Definition 20 Let L be a modal logic. A set of formulas T is said to be an L -expansion of a set of formulas I if $T = Cn_L(I \cup \{\neg\Box\phi : \phi \notin T\})$.

where Cn_L denotes consequence in L . Now we are ready to prove the main theorem of this section.

Theorem 21 Let $\mathcal{M} = (W, R, V)$ be an $S5$ -model, and $T = \{\phi \mid \mathcal{M} \Vdash \phi\}$. Then T is an $wK4f$ -expansion of I if and only if \mathcal{M} is a \mathcal{K} -minimal model of I .

Proof. Assume T is a $wK4f$ -expansion for I . This means that $T = Cn_L[I \cup \{\neg\Box\phi \mid \phi \notin T\}]$, where L stands for $wK4f$. For the contradiction assume \mathcal{M} is not minimal. This means that there is a $wK4f$ -model $\mathcal{N} = (N, S, U)$ such that \mathcal{N} is preferred over \mathcal{M} and $\mathcal{N} \Vdash I$. That \mathcal{N} is preferred over \mathcal{M} means that there is a propositional formula α such that $\mathcal{M} \Vdash \alpha$ while $\mathcal{N} \not\Vdash \alpha$. Now take an arbitrary formula $\psi \notin T$. Since T is an expansion we have that $\Diamond\neg\psi \in T$ hence $\mathcal{M} \Vdash \Diamond\neg\psi$. Hence there is at least one point $w \in W$ with $w \Vdash \neg\psi$ and hence for every point y in the first floor of N we have $y \Vdash \Diamond\neg\psi$ which yields that $\mathcal{N} \Vdash \Diamond\neg\psi$. So we get that $\mathcal{N} \Vdash I \cup \{\neg\Box\phi \mid \phi \notin T\}$ and hence $\mathcal{N} \Vdash T$ which is a contradiction since $\alpha \in T$.

For the other direction assume \mathcal{M} is \mathcal{K} -minimal for I . That $Cn_L[I \cup \{\neg\Box\phi \mid \phi \notin T\}] \subseteq T$ follows directly from the fact that $\mathcal{M} \Vdash I$ and if $\mathcal{M} \not\Vdash \psi$ then there exists at least one point $w \in W$ with $w \Vdash \neg\psi$ and since R is a universal relation, $\mathcal{M} \Vdash \Diamond\neg\psi$.

For the other inclusion we show that for every rooted weakly-transitive and one-step model $\mathcal{N} = (N, S, U)$ the following holds:

$$(*) \quad \mathcal{N} \Vdash Cn_L[I \cup \{\neg\Box\phi \mid \phi \notin T\}] \Rightarrow \mathcal{N} \Vdash T.$$

This by Theorem 3 will imply that $Cn_L[I \cup \{\neg\Box\phi \mid \phi \notin T\}] \vdash T$ in $wK4f$ and, as the left side is closed under consequence, we get that $T \subseteq Cn_L[I \cup \{\neg\Box\phi \mid \phi \notin T\}]$. Now let us prove the star.

Assume $\mathcal{N} \Vdash Cn_L[I \cup \{\neg\Box\phi \mid \phi \notin T\}]$. Note that \mathcal{N} cannot have one irreflexive point as a maximum. This would imply $Ch(N, S) = (i_1, r_1, 1, 0)$, see Theorem 8. Then the irreflexive point does not satisfy $\neg\Box\perp$, hence $\perp \in T$, which is a contradiction as far a T is the theory of \mathcal{M} .

Let us denote the floors of \mathcal{N} by N_1 and N_2 respectively. In case \mathcal{N} is a one-floor frame, $N_2 = \emptyset$. Since \mathcal{K} is weak-cluster closed, there is $\mathcal{N}^* \in \mathcal{K}$ which is either the concatenation of \mathcal{N} and \mathcal{M} or is a cluster substitution of \mathcal{M} in \mathcal{N} . We prove by induction on the complexity of a formula that for every point $w \in N_1$, we have $\mathcal{N}^*, w \Vdash \phi$ iff $\mathcal{N}, w \Vdash \phi$. The only non-trivial case is for formulas of the form $\Box\phi$. Assume $\mathcal{N}, w \Vdash \Box\phi$, then $\phi \in T$. This means that $\mathcal{M} \Vdash \phi$. Now for every point $w' \in N_1$ such that wSw' we have $\mathcal{N}, w' \Vdash \phi$ and hence by the inductive assumption we get that $\mathcal{N}^*, w' \Vdash \phi$. So $\mathcal{N}^*, w \Vdash \Box\phi$.

Conversely assume for some point $w \in N_1$ we have $\mathcal{N}^*, w \Vdash \Box\phi$. By the same argument as in the previous case, for every point $v \in N_1$ such that wSv , $\mathcal{N}, v \Vdash \phi$. Now if \mathcal{N}^* is a concatenation of \mathcal{N} and \mathcal{M} then $N = N_1$, and hence we have $\mathcal{N}, w \Vdash \Box\phi$. In case \mathcal{N}^* is cluster substitution we additionally need to show that for every point $v \in N_2$, $\mathcal{N}, v \Vdash \phi$. From $\mathcal{N}^*, w \Vdash \Box\phi$ we have that $\mathcal{M} \Vdash \phi$ and hence $\mathcal{M} \Vdash \Box\phi$. This implies that $\neg\Box\phi \notin T$ and hence $\mathcal{N} \Vdash \Diamond\Box\phi$. It is not hard to check that this implies

that for every point $v \in N_2$ we have $\mathcal{N}, v \Vdash \phi$. The main point here is that \mathcal{N} cannot have one-irreflexive point as a maximum. Now as $\mathcal{N} \Vdash I$, we have that $\mathcal{N}^* \Vdash I$, hence \mathcal{N}^* is not preferred over \mathcal{M} which implies that $\mathcal{N}^* \Vdash T$. Hence $\mathcal{N} \Vdash T$.

This yields the promised link between $wK4f$ and minimal models in the style of [18, 19]; so non-monotonic $wK4f$ may be a promising first step in the search for logics of minimal belief.

11 Conclusions

Following Tarski and McKinsey there are three natural paths from topology to algebraic semantics for logics. The third path involves derivative algebras and has been explored in particular by Leo Esakia [4, 37, 7] and the Tbilisi group in logic. The first two paths give rise to logics extending intuitionistic logic and to modal $S4$, respectively. In these cases the logics obtained from minimal topological spaces have proved to be highly relevant in AI, for non-monotonic reasoning, logic programming and epistemic logic based on the idea of minimal knowledge. From this point of view, the third path to logics from minimal topological spaces has not previously been investigated. It gives rise to the logic $wK4f$ introduced and characterised in this paper that seems a good starting point for studying the idea of minimal belief, analogous to the minimal knowledge approach of [18, 19] based on $S4F$.

We conclude by mentioning briefly how these two logics, hence their corresponding epistemic concepts, can be formally related. There are well-known embedding relations holding between intuitionistic logic H and $S4$ and between $S4$ and $wK4$. It can be shown that these relations extend to those logics based on minimal topological spaces. In fact we get the following picture

$$\begin{array}{ccccc}
 \text{nonmonotonic} & & EL & \xrightarrow{G} & S4F^* & \xrightarrow{Sp} & wK4f^* \\
 & & \uparrow & & \uparrow & & \uparrow \\
 \text{min-topologies} & & HT & \xrightarrow{G} & S4F & \xrightarrow{Sp} & wK4f \\
 & & \uparrow & & \uparrow & & \uparrow \\
 \text{topologies} & & H & \xrightarrow{G} & S4 & \xrightarrow{Sp} & wK4
 \end{array}$$

Here HT is again the logic of here-and-there, G is the well-known Gödel translation and Sp is known as the splitting translation from modal formulas to modal formulas such that in particular for an atom p , $Sp(\Diamond p) = p \vee \Diamond p$, $Sp(\Box p) = p \wedge \Box p$ (see eg [7]). Then in particular $S4F \vdash \varphi$ iff $wK4f \vdash Sp(\varphi)$. In other words, we obtain the natural interpretation of knowledge as truth together with belief. We plan to elaborate on this in future work and to study how to extend this picture to include the non-monotonic versions of each of the logics at the top of the diagram. Another future topic is to study of the exact relations between non-monotonic $wK4f$ and autoepistemic logic as well as non-monotonic $S4F$.

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